ASPECTS OF CHIRAL INSTABILITY IN HOT NUCLEAR MATTER

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INT Program on Criticality and Chirality:
Novel Phenomena in Heavy-Ion Collisions

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OUTLINE

Motivation: chiral instability drives evolution of gauge fields in chiral matter; applications to QGP, topological materials, astrophysics, cosmology.

Topics:

1. Chiral instability and helicity flow between the chiral matter and the magnetic field. 1702.07329

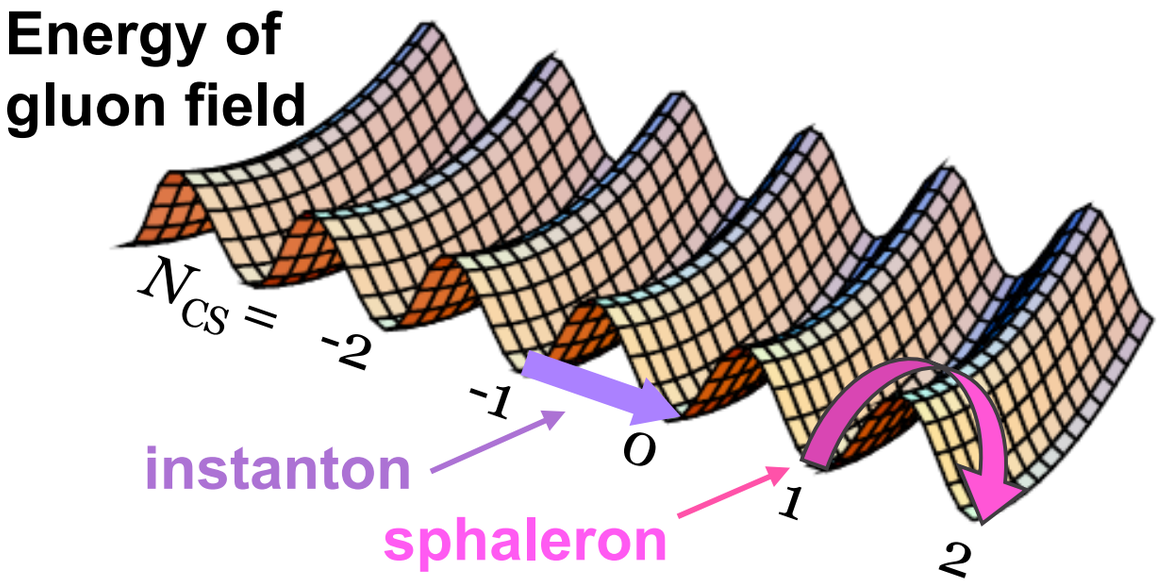
2. Chiral matter with magnetic monopoles. Emergence of superconductivity. 1708.08536

3. Instability of magnetic field in chiral matter with random topological charge (sourced by the color fields). 1911.01357

TOPOLOGY OF THE QCD VACUUM

QCD vacuum is a superposition of states with different topology, characterized by the topological charge density

$$q(x) = \frac{g^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a\mu\nu}(x)$$



Transitions between such states creates local imbalance of chirality.

The transition rate per unit volume is exponentially suppressed at low temperatures, but increases at high temperatures as

$$\Gamma_{\rm sph} \sim (\alpha_s N_c)^5 T^4$$

→ Important phenomenological implications for QGP.

MAXWELL-CHERN-SIMONS THEORY

MCS theory - electrodynamics in topologically nontrivial background $\Delta\mathcal{L} = -\frac{c_A}{4}\theta F_{\mu\nu}\tilde{F}^{\mu\nu}$

Sikivie (1984), Wilczek (1987)

Equations of motion of electromagnetic field

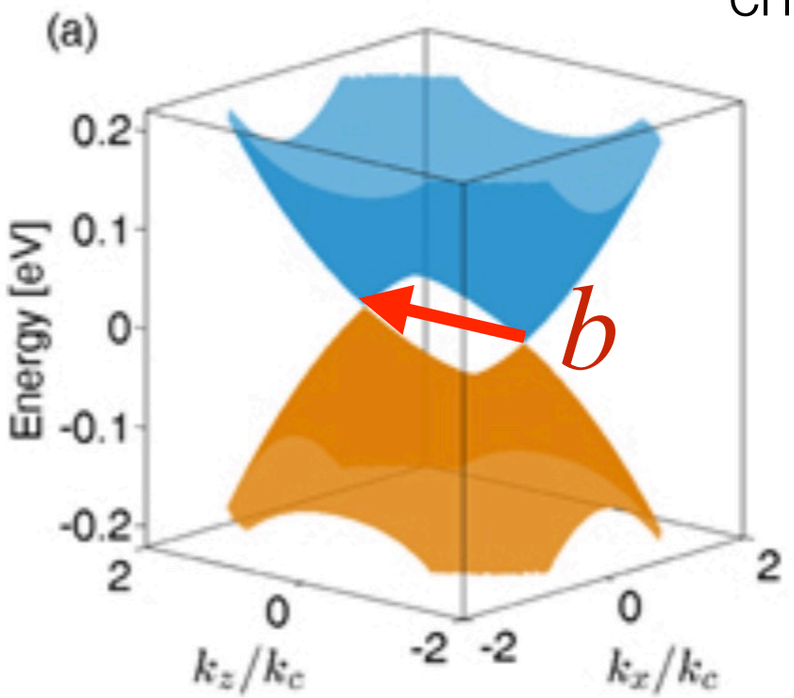
$$\partial_\mu F^{\mu\nu} = j^\nu - c_A \tilde{F}^{\mu\nu}\partial_\mu\theta\,,$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0\,.$$

Often used notations:

$\sigma_\chi = c_A\dot{\theta}$

"chiral conductivity" axial chemical potential μ_5



$$\boldsymbol{b} = c_A\boldsymbol{\nabla}\theta.$$

Separation of Weyl nodes in Weyl semimetals

ANOMALOUS CURRENTS IN QGP

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \partial_t \boldsymbol{E} + \boldsymbol{j} + c_A(\partial_t \theta\, \boldsymbol{B} + \boldsymbol{\nabla}\theta \times \boldsymbol{E})$$

The anomalous currents

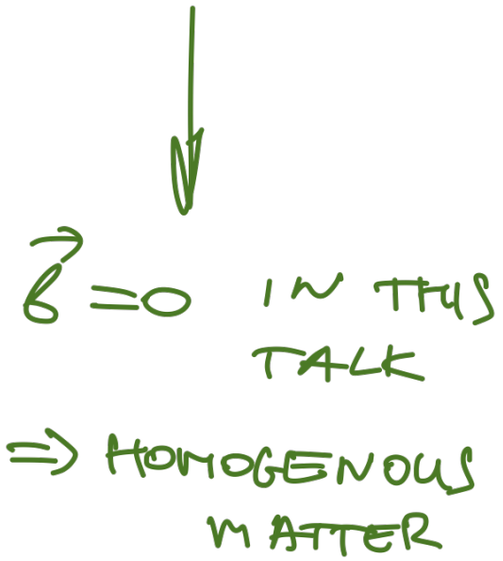
$$\boldsymbol{j}_{\mathrm{CME}} = \sigma_\chi \boldsymbol{B}$$

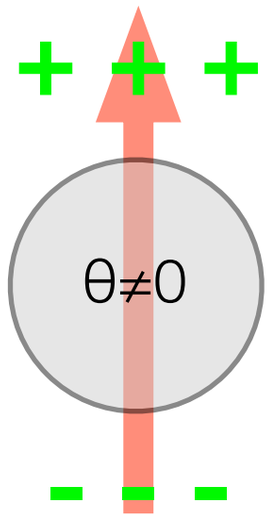
P-odd, T-odd

P-even, T-odd

$$\boldsymbol{j}_{\mathrm{AHE}} = \boldsymbol{b} \times \boldsymbol{E}$$

Anomalous Hall Effect





External magnetic field drives the charge separation.

Breaks Parity!

Chiral magnetic effect

Kharzeev, McLerran, Warringa (2008)

INSTABILITY AT CONSTANT σ_χ

Far away from any sources, Maxwell equations in momentum space read

$$\boldsymbol{k}\cdot\boldsymbol{B}_{\omega,\boldsymbol{k}}=0\,,$$

$$\epsilon\boldsymbol{k}\cdot\boldsymbol{E}_{\omega,\boldsymbol{k}}=0\,,$$

$$\boldsymbol{k}\times\boldsymbol{E}_{\omega,\boldsymbol{k}}=\omega\boldsymbol{B}_{\omega,\boldsymbol{k}}\,,$$

$$\boldsymbol{k}\times\boldsymbol{B}_{\omega,\boldsymbol{k}}=-\omega\epsilon\boldsymbol{E}_{\omega,\boldsymbol{k}}-i\sigma_\chi\boldsymbol{B}_{\omega,\boldsymbol{k}}$$

Electromagnetic waves have the dispersion relation

$$[\omega(\omega+i\sigma)-\boldsymbol{k}^2]^2=\sigma_\chi^2\boldsymbol{k}^2$$

Four solutions: $\omega_{\lambda_1,\lambda_2}=-\dfrac{i\sigma}{2}+\lambda_1\sqrt{k^2+\lambda_2\sigma_\chi k-\sigma^2/4}$ $\qquad\qquad\qquad

CHIRAL INSTABILITY

Limits on a Lorentz- and parity-violating modification of electrodynamics

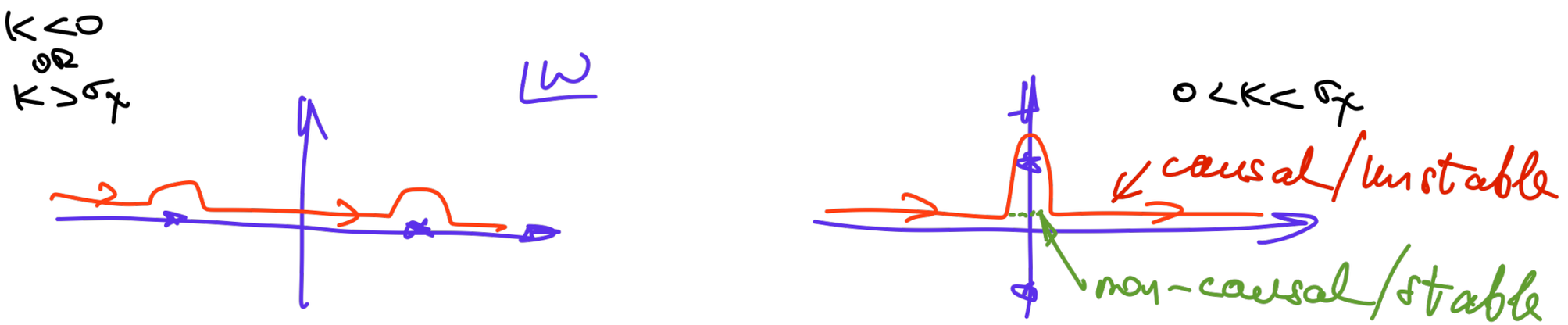
Sean M. Carroll and George B. Field
Harvard-Smithsonian Center for Astrophysics, Cambridge, Massachusetts 02138

Roman Jackiw*
Department of Physics, Columbia University, New York, New York 10027
(Received 5 September 1989)

Photon propagator $G^{ij}(t,\mathbf{r})=[(\delta^{ij}-\partial_i\partial_j/\nabla^2)\Box+m\epsilon^{ijk}\partial_k]g(t,\mathbf{r})$

$$g(t,r)=\frac{2\sin\frac{m}{2}r}{\pi m r}\left[\theta(t)\int_0^\infty dz\frac{\cos r\left[\frac{m^2}{4}+z^2\right]^{1/2}}{\left[\frac{m^2}{4}+z^2\right]^{1/2}}\sin tz-\tfrac{1}{2}\int_0^{m/2}dz\frac{\cos r\left[\frac{m^2}{4}-z^2\right]^{1/2}}{\left[\frac{m^2}{4}-z^2\right]^{1/2}}e^{-|t|z}\right].$$

$m=\sigma_\chi$



A very partial list of other references: Redlich, Wijewardhana(1985), Rubakov (1986), Joyce, Shaposhnikov (1987), Boyarsky at al (2012), Sadofyev, Zakharov et al (2013), Akamatsu,Yamamoto(2013), Hirono, Kharzev, Yin (2015), Manuel, Torres-Rincon(2015), Buividovich, Ulybyshev (2016), Kaplan, Reddy, Sen (2016).

THE FASTEST GROWING STATE

$$-\nabla^2 \boldsymbol{A} = -\partial_t^2 \boldsymbol{A} + \boldsymbol{j} + \sigma_\chi(t)\boldsymbol{\nabla} \times \boldsymbol{A}$$

Expand in the circularly polarized plane wave basis

$$\boldsymbol{A} = \sum_{\boldsymbol{k},\lambda} [a_{\boldsymbol{k}\lambda}(t)\boldsymbol{W}_{\boldsymbol{k}'\lambda'}(\boldsymbol{x}) + a^*_{\boldsymbol{k}'\lambda'}(t)\boldsymbol{W}^*_{\boldsymbol{k}'\lambda'}(\boldsymbol{x})]$$

Plane wave amplitude satisfies $k^2 a_{\boldsymbol{k}\lambda} = -\ddot{a}_{\boldsymbol{k}\lambda} - \sigma\dot{a}_{\boldsymbol{k}\lambda} + \sigma_\chi(t)\lambda k\, a_{\boldsymbol{k}\lambda}$

Seeking solution in the form $a_{\boldsymbol{k}\lambda} = e^{-i\int_0^t \omega_{k\lambda}(t')dt'}$

Assume that $\omega_{k\lambda}(t)$ is a slowly varying function of time

$$\omega_{k\lambda}(t) = \left[-\frac{i\sigma}{2} + \lambda_1\sqrt{k^2 - \sigma_\chi\lambda k - \sigma^2/4}\right] \qquad \lambda_1 = \pm 1$$

THE FASTEST GROWING STATE

The fastest growing state has largest imaginary $\omega_{k\lambda}(t)$ which happens when

$$k_0 = \frac{\sigma_\chi \lambda}{2} \qquad \text{for which} \qquad \omega_0(t) = -\frac{i\sigma}{2} + \frac{i}{2}\sqrt{\sigma^2 + \sigma_\chi^2(t)}$$

(suppose $\sigma_\chi > 0 \quad \lambda = 1$)

The fastest growing state: $a_0(t) = e^{\gamma(t)/2}$ $\qquad \gamma(t) = \int_0^t \left[\sqrt{\sigma^2 + \sigma_\chi^2(t')} - \sigma\right] dt'$

Approximately: $\boldsymbol{A}(\boldsymbol{r}, t) \approx a_0(t)\boldsymbol{W}_{\boldsymbol{k}_0+}(\boldsymbol{r})$ $\qquad \boldsymbol{W}_{\boldsymbol{k}\lambda}(\boldsymbol{x}) = \frac{\boldsymbol{\epsilon}^\lambda}{\sqrt{2kV}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$

(Note that $\boldsymbol{W}_{\boldsymbol{k}_0+}(\boldsymbol{r})$ depends on time)

👉 We got ***A*** with growth rate *dynamically* depending on the medium helicity.

ENERGY AND HELICITY

Energy stored in the EM field: $\mathcal{E}_{\rm em} = \frac{1}{2}\int (\boldsymbol{E}^2 + \boldsymbol{B}^2) d^3x = \sum_{\boldsymbol{k},\lambda} \frac{1}{2k} \left(|\dot{a}_{\boldsymbol{k}\lambda}|^2 + |a_{\boldsymbol{k}\lambda}|^2 k^2 \right)$

$$= \frac{1}{2\sigma_\chi} \left(\sigma_\chi^2 + \sigma^2 - \sigma\sqrt{\sigma^2 + \sigma_\chi^2} \right) e^\gamma$$

Magnetic field helicity $\mathcal{H}_{\rm em} = \int \boldsymbol{A}\cdot\boldsymbol{B}\, d^3x = \sum_{\boldsymbol{k},\lambda} \lambda |a_{\boldsymbol{k}\lambda}|^2 = e^\gamma$

Energy loss due to Ohm's currents $Q = \int \boldsymbol{j}\cdot\boldsymbol{E}\, d^3x = \sigma \int \boldsymbol{E}^2\, d^3x = \sigma \sum_{\boldsymbol{k},\lambda} \frac{1}{k} |\dot{a}_{\boldsymbol{k}\lambda}|^2$

$$= \frac{\sigma}{2\sigma_\chi} \left(\sqrt{\sigma^2 + \sigma_\chi^2} - \sigma \right)^2 e^\gamma$$

Maxwell equations can be used to derive the energy balance equation

$$\frac{1}{2}\partial_t \int (\boldsymbol{E}^2 + \boldsymbol{B}^2) d^3x + \int \boldsymbol{j}\cdot\boldsymbol{E}\, d^3x + \sigma_\chi(t) \int \boldsymbol{B}\cdot\boldsymbol{E}\, d^3x = 0$$

Up to a surface term $\partial_t \mathcal{H}_{\rm em} = -2\int \boldsymbol{E}\cdot\boldsymbol{B}\, d^3x \quad \Rightarrow \quad \partial_t \mathcal{E}_{\rm em} + Q - \frac{1}{2}\sigma_\chi(t)\, \partial_t \mathcal{H}_{\rm em} = 0$

FGS satisfies this condition for any chiral conductivity. *Thus, energy conservation does not prevent instability.*

HELICITY CONSERVATION

Chiral anomaly $\partial_\mu j^\mu_A = c_A \boldsymbol{E}\cdot\boldsymbol{B}$

In a homogeneous medium $\boldsymbol{\nabla}\cdot\boldsymbol{j}_A = \sigma_\chi \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \quad \Rightarrow \quad \dot{n}_A = c_A \boldsymbol{E}\cdot\boldsymbol{B}$

Averaging over volume $\partial_t \langle n_A\rangle = \dfrac{c_A}{V}\int \boldsymbol{E}\cdot\boldsymbol{B}\, d^3x = -\dfrac{c_A}{2V}\partial_t \mathcal{H}_{\rm em}$

Integrate over time $\Rightarrow$

$$\frac{2V}{c_A}\langle n_A\rangle + \mathcal{H}_{\rm em} = \mathcal{H}_{\rm tot}$$

Magnetic helicity depends on $\sigma_\chi(t)$ rather than on $\langle n_A\rangle$ $\Rightarrow$ need equation of state

In hot medium $\langle n_A\rangle = \chi\,\mu_5$ (recall: $\sigma_\chi = c_A\mu_5$)

Equation of state $\sigma_\chi(t) = \dfrac{c_A}{\chi}\langle n_A(t)\rangle$

EVOLUTION OF CHIRAL CONDUCTIVITY

Helicity conservation

$$\frac{\sigma_\chi(t)}{\alpha} = 1 - \frac{\mathcal{H}_{\rm em}(t)}{\mathcal{H}_{\rm tot}}$$

Helicity flows between the magnetic field and medium. Total helicity is conserved.

Fraction of the total helicity in plasma

Fraction of the total helicity in the field

Vector potential is normalized so that

$$\mathcal{H}_{\rm em}(0) = 1$$

Using the FGS ansatz in the helicity conservation equation:

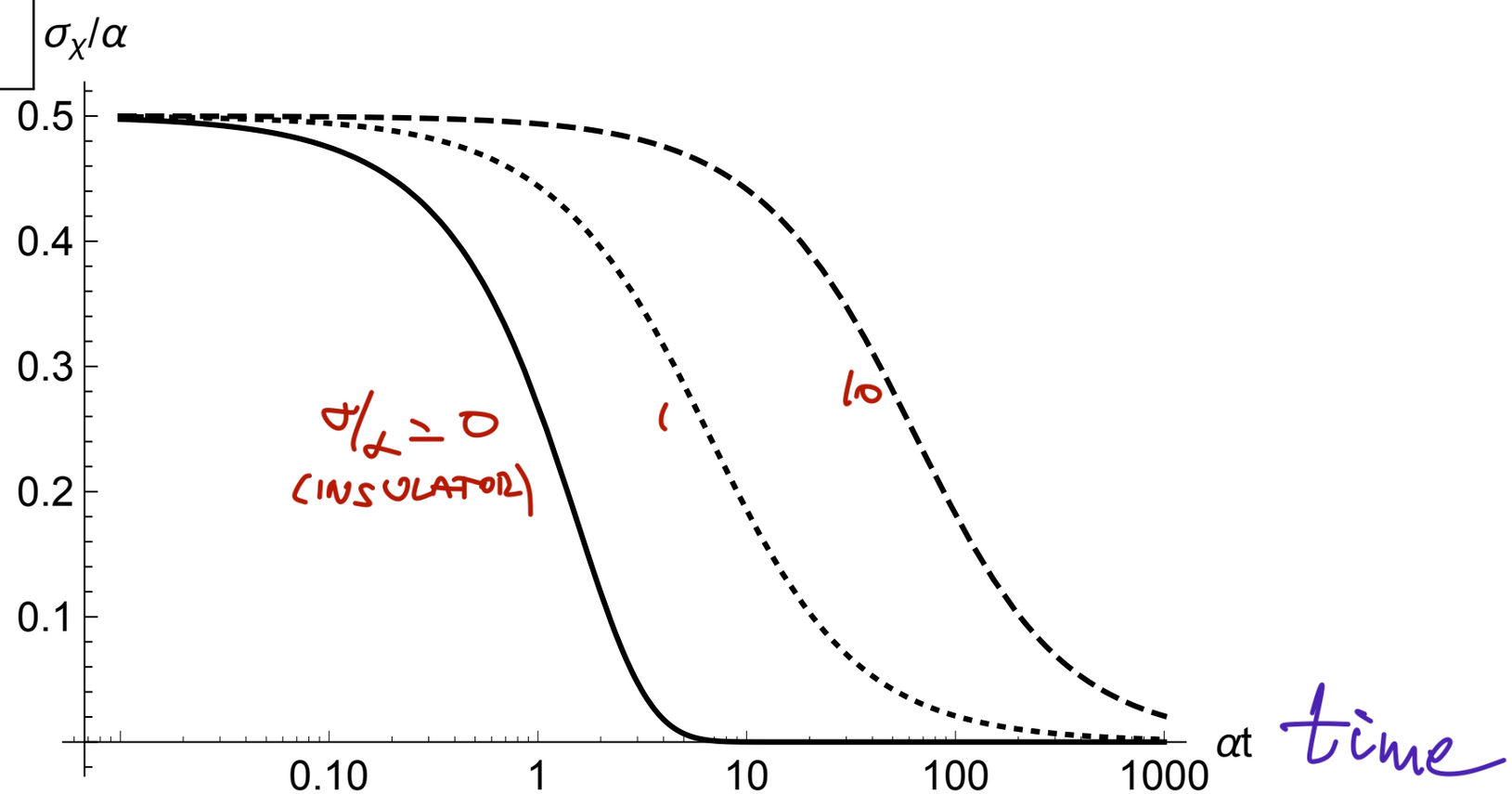
$$\dot{\sigma}_\chi = -\left(\sqrt{\sigma^2 + \sigma_\chi^2} - \sigma\right)(\alpha - \sigma_\chi)$$

Characteristic energy scale

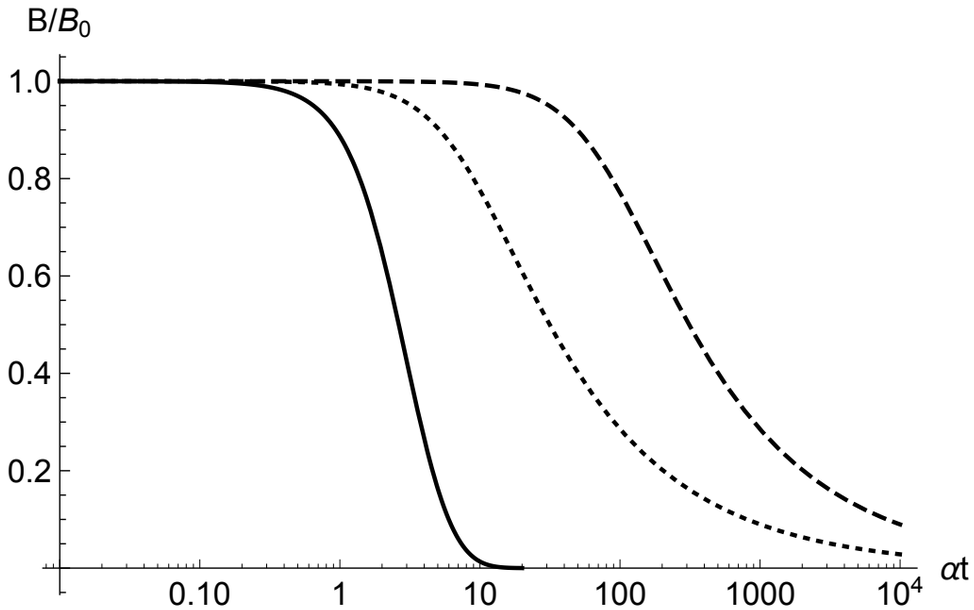
$$\alpha = \mathcal{H}_{\rm tot} c_A^2/(2V\chi)$$

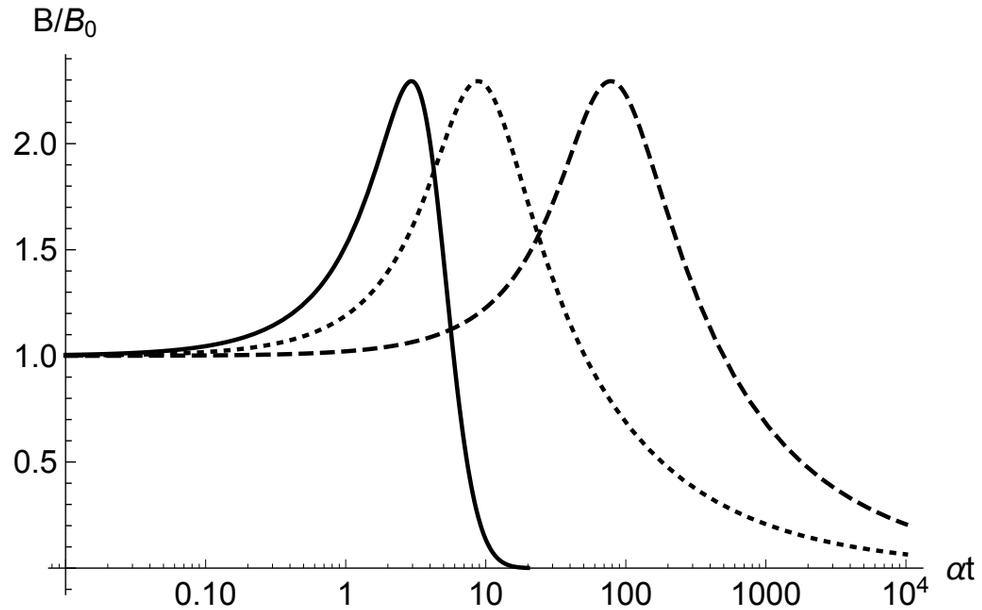
EVOLUTION OF CHIRAL CONDUCTIVITY

Fraction of the total helicity in plasma



EVOLUTION OF CHIRAL CONDUCTIVITY



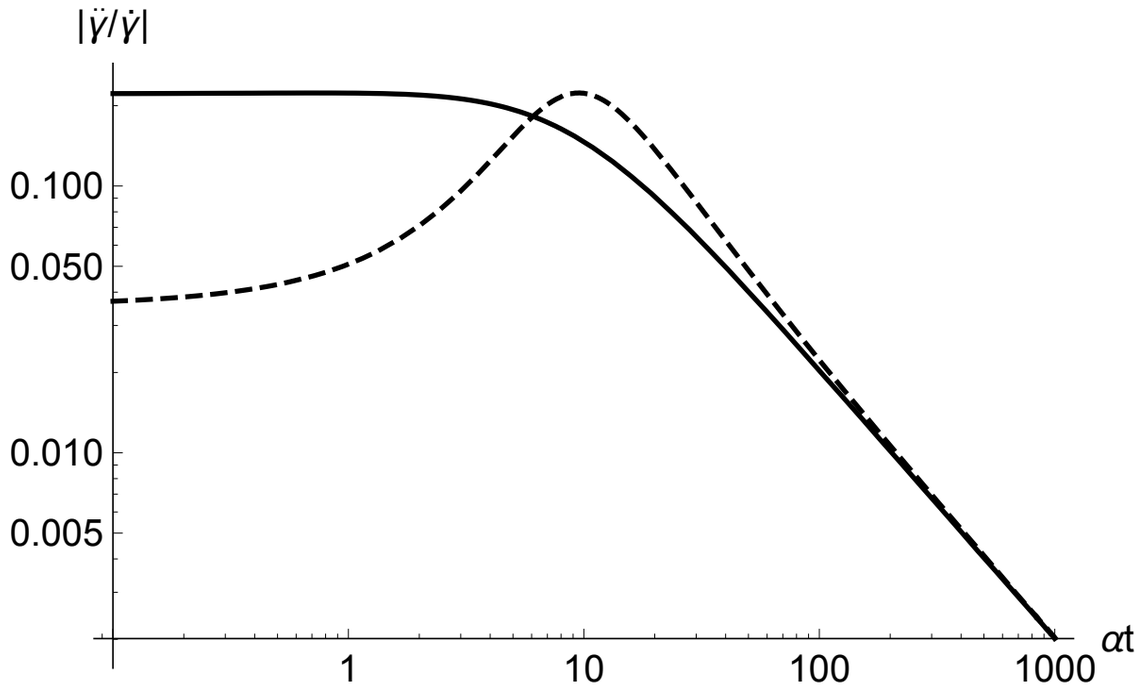


50 % OF INITAL HELICITY IS IN MEDIUM

95 % OF INITAL HELICITY IS IN MEDIUM

Magnetic field develops a maximum only if the initial helicity in medium >50%

VALIDITY OF ADIABATIC APPROXIMATION



Works very well!

SUMMARY 1

1. Chiral instability drives evolution of magnetic field in chiral matter.

2. Energy conservation does not tame the magnetic field instability, but the chiral anomaly does.

3. Magnetic field can grow only if sufficient helicity is initially in the medium.

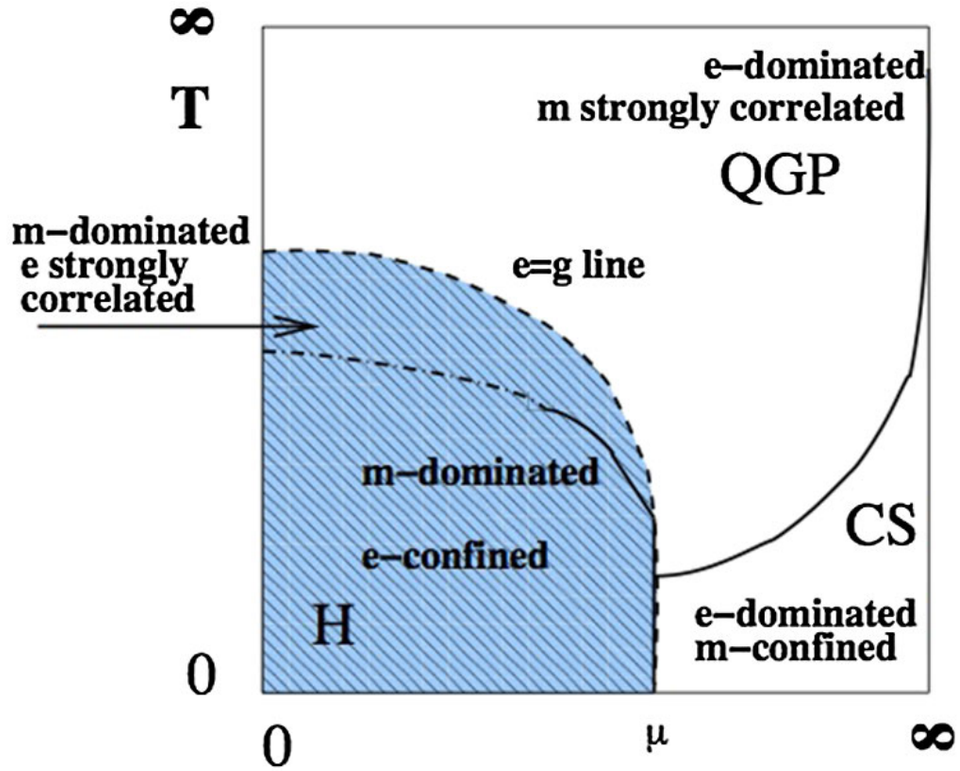
MAGNETIC MONOPOLES AND THE INSTABILITY

Motivation:

Magnetic monopoles at T=0: dual superconductor, color confinement.

The condensate may not melt away at T_c

⇒ Important part of QGP dynamics



Shuryak, 2009

Chiral evolution with magnetic monopoles (Abelian model).

MAXWELL-CHERN-SIMONS WITH MONOPOLES

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=0\,,\qquad -\boldsymbol{\nabla}\times\boldsymbol{E}=\partial_t\boldsymbol{B}+\boldsymbol{j}_m$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=0\,,\qquad \boldsymbol{\nabla}\times\boldsymbol{B}=\partial_t\boldsymbol{E}+\boldsymbol{j}_e+\sigma_\chi\boldsymbol{B}\,,$$

Assume linear response: $\boldsymbol{j}_e=\sigma_e\boldsymbol{E},\ \boldsymbol{j}_m=\sigma_m\boldsymbol{B},$

$$-\nabla^2\boldsymbol{B}+\partial_t^2\boldsymbol{B}=-(\sigma_e+\sigma_m)\partial_t\boldsymbol{B}-\sigma_e\sigma_m\boldsymbol{B}+\sigma_\chi(t)\boldsymbol{\nabla}\times\boldsymbol{B}$$

Chiral anomaly equation $\partial_t\sigma_\chi=c_A^2/(\chi V)\int\boldsymbol{E}\cdot\boldsymbol{B}\,d^3x\,,$

👉 $$\beta^{-1}\partial_t\sigma_\chi=-\partial_t\mathcal{H}_{\rm em}-2\sigma_m\mathcal{H}_{\rm em}$$

⇒ Magnetic current dissipates magnetic helicity

and energy because is σ_m is T-odd

SUPERCONDUCTING PHASE

Define "vector potential" $\boldsymbol{E} = -\partial_t \boldsymbol{A} - \sigma_m \boldsymbol{A}$ $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}.$ $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$

$\boldsymbol{E} = -\partial_t \boldsymbol{A} - \sigma_m \boldsymbol{A}$ reminiscent of the London theory of superconductivity

Note, however, that A does not transform under the gauge transform as a vector potential has to.

Define the "normal" and "super" fields $\boldsymbol{E}_n = -\partial_t \boldsymbol{A}\,,$ $\boldsymbol{E}_s = -\sigma_m \boldsymbol{A}\,.$

and currents $\boldsymbol{j}_n = \sigma_e \boldsymbol{E}_n\,,$ $\boldsymbol{j}_s = \sigma_e \boldsymbol{E}_s = -\sigma_e \sigma_m \boldsymbol{A}\,.$

charge conservation $\boldsymbol{\nabla} \cdot \boldsymbol{j}_n = \boldsymbol{\nabla} \cdot \boldsymbol{j}_s = 0.$

London equations: $\boldsymbol{\nabla} \times \boldsymbol{j}_s = -\sigma_e \sigma_m \boldsymbol{B},$ $\partial_t \boldsymbol{j}_s = +\sigma_e \sigma_m \boldsymbol{E}_n,$

$\Rightarrow$ when $\boldsymbol{E}_\mathrm{n}$=0, $\boldsymbol{j}_\mathrm{s}$ is finite

MEISSNER EFFECT

Stationary limit: $\nabla^2 \boldsymbol{B} = \sigma_e \sigma_m \boldsymbol{B} - \sigma_\chi \boldsymbol{\nabla} \times \boldsymbol{B}\,,$

$$\nabla^2 \boldsymbol{E}_s = \sigma_e \sigma_m \boldsymbol{E}_s - \sigma_\chi \boldsymbol{\nabla} \times \boldsymbol{E}_s$$

Spectrum:

$$k = \frac{\lambda \sigma_\chi}{2} \pm \sqrt{\frac{1}{4}\sigma_\chi^2 - \sigma_e \sigma_m}.$$

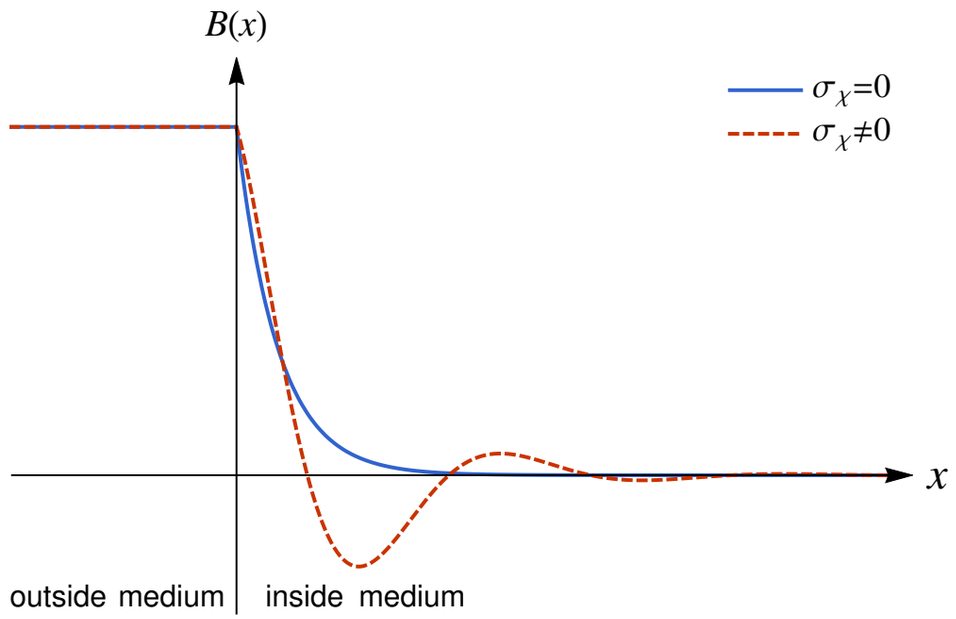


FIG. 1. Meissner effect in a chiral medium.

Meissner effect appears if $\sigma_\chi^2 < 4\sigma_e \sigma_m$

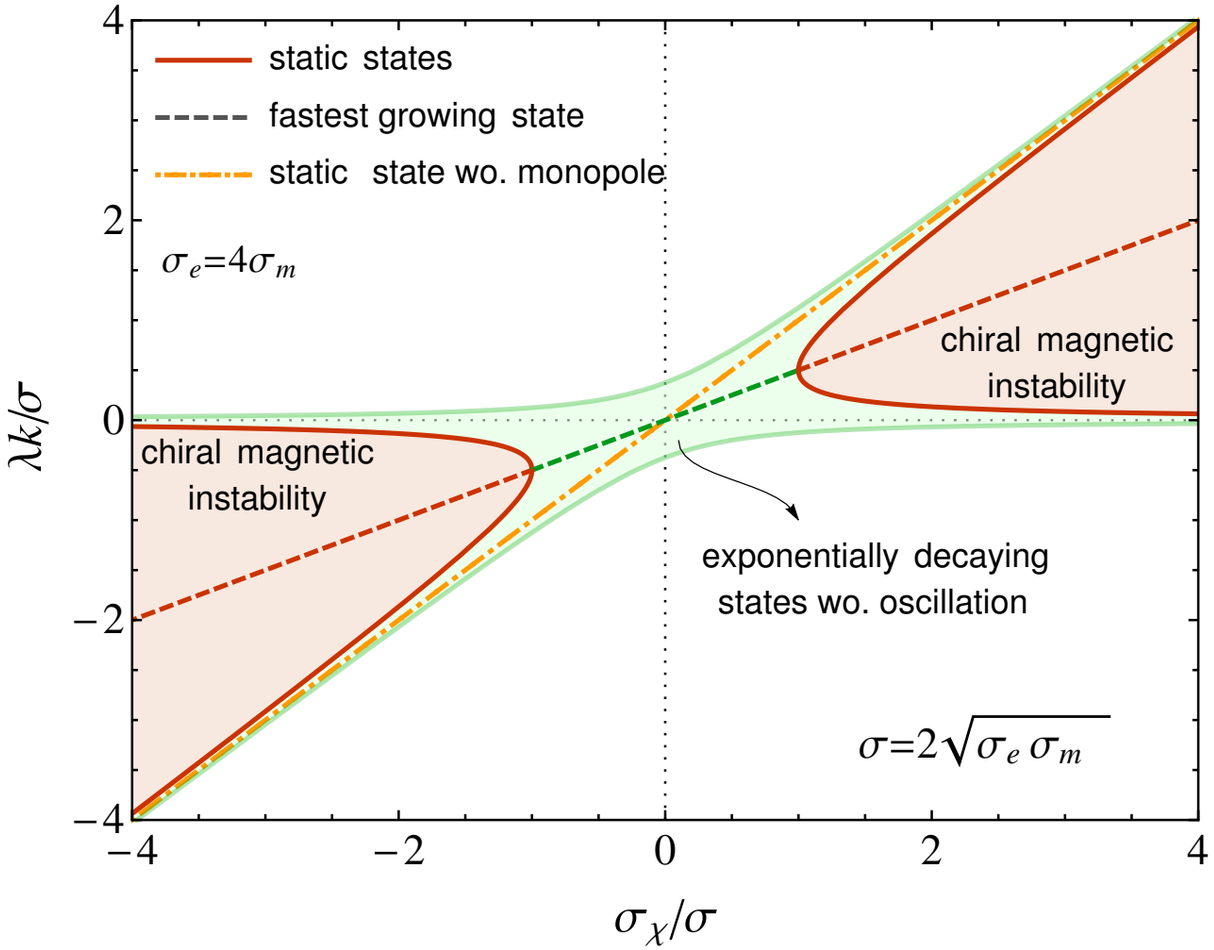
London penetration depth $L = (\sigma_e \sigma_m - \sigma_\chi^2/4)^{-1/2}$

Same is true of electric field. Moreover, one can introduce $\boldsymbol{A}$ such that $\boldsymbol{E}$=curl $\boldsymbol{A}$.
Everything is dual!

ADIABATIC SOLUTION OF MCS WITH MONOPOLES

$$-\nabla^2 \boldsymbol{B} + \partial_t^2 \boldsymbol{B} = -(\sigma_e + \sigma_m)\partial_t \boldsymbol{B} - \sigma_e \sigma_m \boldsymbol{B} + \sigma_\chi(t) \boldsymbol{\nabla} \times \boldsymbol{B}$$

Dispersion: $\omega_{k\lambda}(t) = -\frac{i(\sigma_e + \sigma_m)}{2} + \lambda_1 \frac{i}{2}\sqrt{(\sigma_e + \sigma_m)^2 + 4(\sigma_\chi \lambda k - \sigma_e \sigma_m - k^2)}$ $\qquad \lambda_1 = \pm 1$



Instability (exponential increase of $\mathcal{H}$) occurs for $Im\ \omega > 0$:

$$\frac{\sigma_\chi}{2} - \frac{1}{2}\sqrt{\sigma_\chi^2 - 4\sigma_e\sigma_m} \le k \le \frac{\sigma_\chi}{2} + \frac{1}{2}\sqrt{\sigma_\chi^2 - 4\sigma_e\sigma_m}\,.$$

$\Rightarrow$ only possible if $\sigma_\chi^2 > 4\sigma_e\sigma_m$.

👉 chiral instability is incompatible with the Meissner effect

BTW, the Fastest Growing State model captures the essential qualitative features.

FASTEST GROWING STATE WITH MONOPOLES

Use FGS model to illustrate the general features the chiral evolution.

$$\omega_{k\lambda}(t) = -\frac{i(\sigma_e+\sigma_m)}{2} + \lambda_1 \frac{i}{2}\sqrt{(\sigma_e+\sigma_m)^2 + 4(\sigma_\chi \lambda k - \sigma_e\sigma_m - k^2)}$$

The growth rate is largest for $k_\star = \dfrac{\sigma_\chi \lambda}{2}$,

Amplitude $a_\star(t) = a_\star(0) e^{\frac{1}{2}\gamma(t)}$ $\gamma(t) = \displaystyle\int_0^t \left[\sqrt{(\sigma_e-\sigma_m)^2 + \sigma_\chi^2(t')} - (\sigma_e+\sigma_m)\right] dt'$

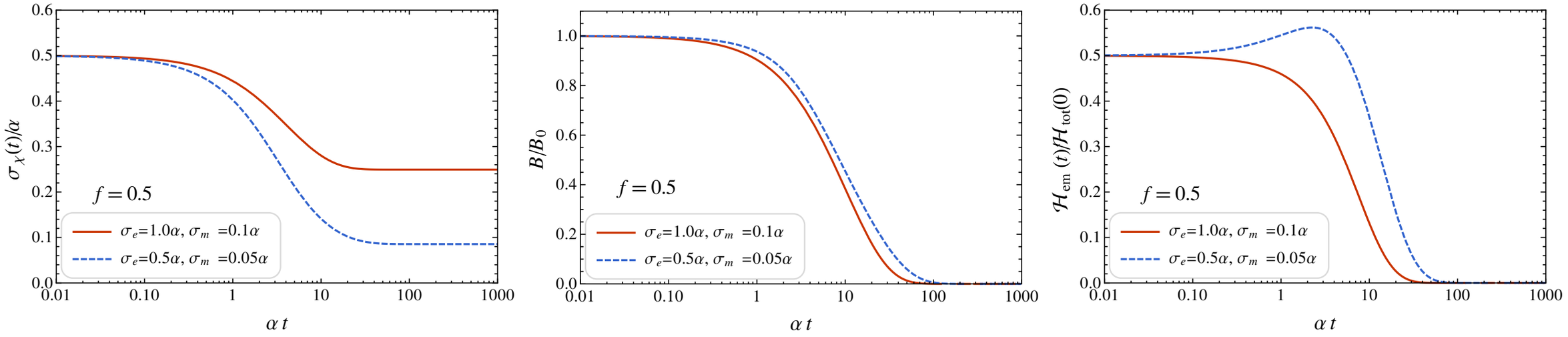
Magnetic helicity $\mathcal{H}_{\rm em}(t) = f\mathcal{H}_{\rm tot}(0) e^{\gamma(t)}$ f is the initial fraction of the helicity stored in B

One can derive a closed equation governing time-evolution of the chiral conductivity

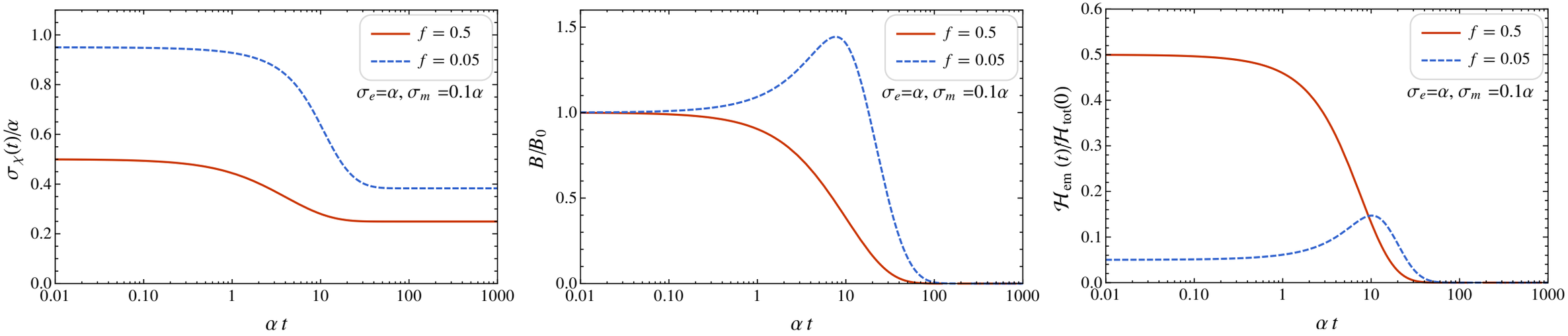
$$\dot\sigma_\chi = -\left[f + F(1-f) - F(\sigma_\chi)\right]\left(\sqrt{(\sigma_e-\sigma_m)^2+\sigma_\chi^2} - \sigma_e + \sigma_m\right)$$

$$F(\sigma_\chi) = \frac{1}{\sigma_\chi}\left\{\sigma_\chi^2 + 2\sigma_m\left[\sqrt{\sigma_\chi^2 + (\sigma_e-\sigma_m)^2} + \sigma_e - \sigma_m\right] - 2\sigma_\chi\sigma_m \ln\left[\sigma_\chi + \sqrt{\sigma_\chi^2 + (\sigma_e-\sigma_m)^2}\right]\right\},$$

EXAMPLES OF EVOLUTION WITH MONOPOLES



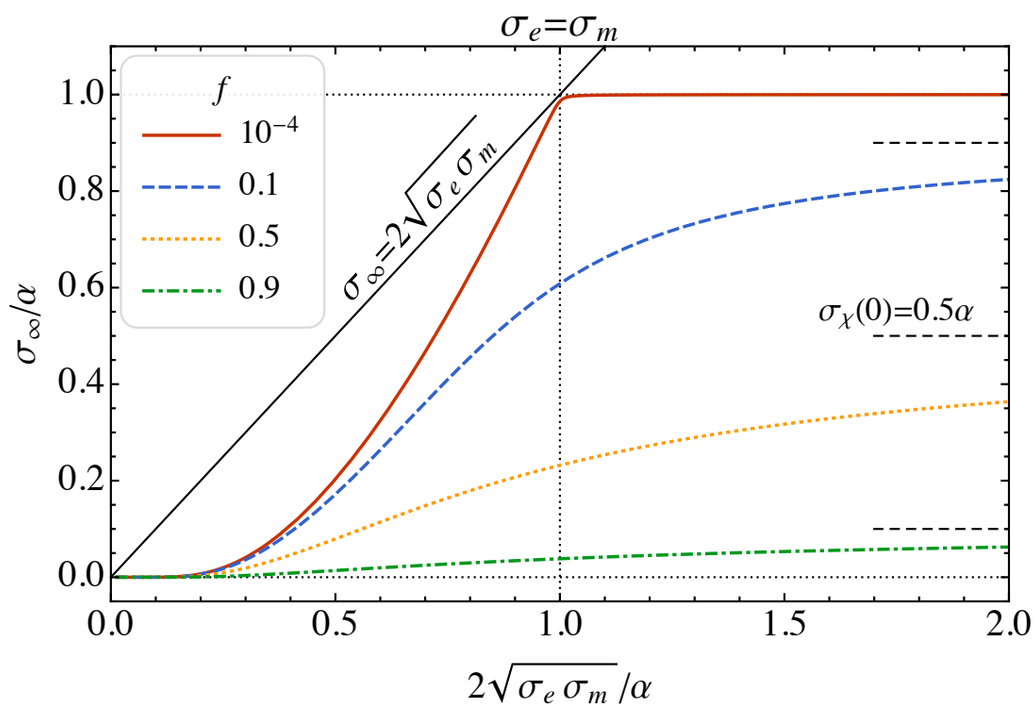
(a) $f = 0.5,\ \sigma_e = 10\sigma_m$



(c) $\sigma_e = 1\alpha, \sigma_m = 0.1\alpha$

Notice that σ_χ does not vanish at $t \to \infty$ unlike no-monopole case. This is because part of magnetic helicity leaks out and the evolution stops when the system runs out of the magnetic helicity, but the medium helicity is finite.

FINAL VALUES OF CHIRAL CONDUCTIVITY



At $t\to\infty$ σ_χ approaches a constant that satisfies

$$\sigma_\chi^2(t\to\infty) < 4\sigma_e\sigma_m$$

😲 Final state is stable and exhibits the Meisner effect!

Stability = exponential decay of small magnetic helicity fluctuations

WHEN MAGNETIC HELICITY IS STABLE

Using FGS one can derive a more stringent stability condition.

Instability of magnetic helicity = period of growth → stability condition is $\dot{\mathcal{H}}_{\rm em} < 0$.

$$\mathcal{H}_{\rm em} = \mathcal{H}_{\rm tot}(0)\left[f + F(1-f) - F(\sigma_\chi)\right].$$

→ $F'(\sigma_\chi)\dot{\sigma}_\chi > 0.$

$$\dot{\sigma}_\chi = -\left[f + F(1-f) - F(\sigma_\chi)\right]\left(\sqrt{(\sigma_e-\sigma_m)^2+\sigma_\chi^2} - \sigma_e + \sigma_m\right) \qquad \rightarrow \quad \dot{\sigma}_\chi < 0$$

→ $F'(\sigma_\chi) < 0$

$$F(\sigma_\chi) = \frac{1}{\sigma_\chi}\left\{\sigma_\chi^2 + 2\sigma_m\left[\sqrt{\sigma_\chi^2+(\sigma_e-\sigma_m)^2} + \sigma_e - \sigma_m\right] - 2\sigma_\chi\sigma_m \ln\left[\sigma_\chi + \sqrt{\sigma_\chi^2+(\sigma_e-\sigma_m)^2}\right]\right\},$$

→ $\sigma_\chi^2 \leq 4\sigma_e\sigma_m$

In particular,

$$\boxed{\sigma_\chi^2(0) \leq 4\sigma_e\sigma_m\,.}$$

SUMMARY 2

1. There is a superconducting phase in MCS with linear responses; Meissner effect for both $\boldsymbol{E}$ and $\boldsymbol{B}$

2. Chiral evolution always ends up in a superconducting state satisfying $\sigma_\chi^2 < 4\sigma_e\sigma_m$

3. A superconducting state is always stable: a small fluctuation of magnetic helicity decays exponentially with time.

4. Superfluid component may explain very low viscosity of QGP

Close CMP analogue: spin ice, but has quadratic response

🤔 Interesting: what is the role of the Dirac quantization $eg=N/2$?

TIME-VARIATION OF TOPOLOGICAL CHARGE

❖ Early times $t \ll \tau_c$ ⇒ σ_χ is adiabatic

where $\tau_c \sim 1/(g^4 T)$ sphaleron transition time

❖ By the time t=τ_c the chiral instability becomes $\exp(\sigma_\chi \tau_c)$

Since $\sigma_\chi \sim e^2 \mu_5$ ⇒ $\sigma_\chi \tau_c \ll 1$ ⇒

the topological charge changes by the time the instability fully develops.

❖ How does magnetic field evolve at $t \gg \tau_c$?

Model: the chiral conductivity is a stochastic process with

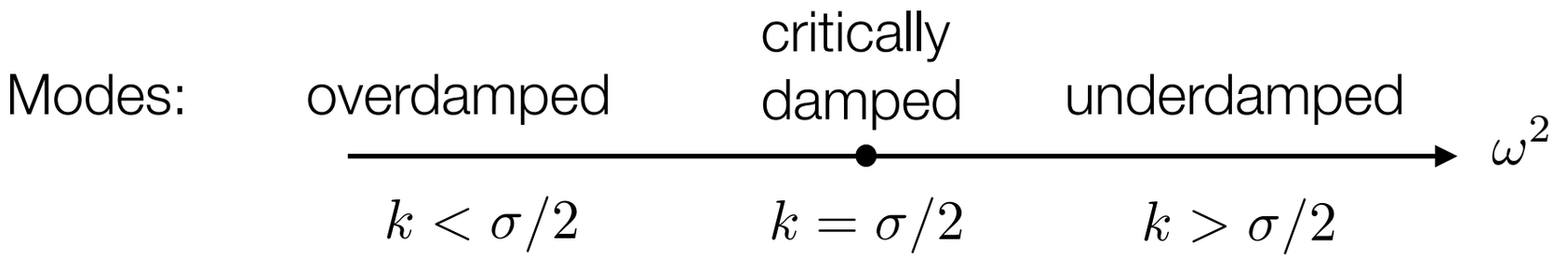
$\langle \sigma_\chi \rangle = 0$ $\Sigma_\chi = \sqrt{\langle \sigma_\chi^2 \rangle} = c_A \mu_5$ $\langle \sigma_\chi(t) \sigma_\chi(t-\tau) \rangle \neq 0$ when $t < \tau_c$

HARMONIC OSCILLATOR WITH RANDOM FREQUENCY

$$\boldsymbol{A} = a_{\boldsymbol{k}\lambda}(t)\boldsymbol{\epsilon}_\lambda e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \qquad \ddot{a}_{\boldsymbol{k}\lambda} + \sigma\dot{a}_{\boldsymbol{k}\lambda} + k(k - \lambda\sigma_\chi)a_{\boldsymbol{k}\lambda} = 0$$

$$x = a_{\boldsymbol{k}\lambda}e^{\sigma t/2} \quad \Rightarrow \quad \ddot{x}(t) + \omega^2[1 + \alpha\xi(t)]x(t) = 0$$

$$\omega^2 = k^2 - \frac{\sigma^2}{4}\,, \qquad \alpha = -\frac{\lambda k}{\omega^2}\Sigma_\chi\,, \qquad \xi = \frac{\sigma_\chi}{\Sigma_\chi}\,, \qquad \Sigma_\chi = \sqrt{\langle\sigma_\chi^2\rangle}$$



GENERAL THEORY

Harmonic oscillator with random frequency belongs to the class of linear stochastic equations

$$\frac{du(t')}{dt'} = [A_0 + \alpha\xi(t')B]u(t') \qquad t' = \omega t$$

It can be converted to an ordinary integro-differential equation at $t \gg \tau_c$

$$\frac{d\langle u(t')\rangle}{dt'} = \left\{A_0 + \alpha^2 \int_0^\infty \langle \xi(t')\xi(t'-\tau')\rangle \, Be^{A_0\tau'} Be^{-A_0\tau'} d\tau'\right\} \langle u(t')\rangle$$

Van Kampen (1975)

provided that $\alpha \ll 1$ i.e. the fluctuating term is a perturbation.

In our case this condition is satisfied if ω is not too small, i.e. k is not too close to σ/2

Expanding $k = \sigma/2 + \delta k \;\Rightarrow\; |\delta k| \gg \frac{\sigma}{4}(\Sigma_\chi \tau_c)^2 \;\Rightarrow$

Holds everywhere except a very narrow interval around σ/2.

EVOLUTION OF <AMPLITUDE>

Equation for the first moments $u = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$ $A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$

$$\frac{d^2\langle x\rangle}{dt'^2} + \frac{1}{2}\alpha^2 c_2 \frac{d\langle x\rangle}{dt'} + \left(1 - \frac{1}{2}\alpha^2 c_1\right)\langle x\rangle = 0$$

$$c_1 = \int_0^\infty \langle \xi(t')\xi(t'-\tau')\rangle \sin(2\tau')d\tau' ,$$

$$c_2 = \int_0^\infty \langle \xi(t')\xi(t'-\tau')\rangle \left[1-\cos(2\tau')\right]d\tau'$$

Solution: $$\langle x(t)\rangle_\pm = \exp\left\{\pm i\omega t - \frac{\alpha^2}{4}(c_2 \pm i c_1)\omega t\right\}$$

$$\Rightarrow \quad \langle a_{\boldsymbol{k}\lambda}(t)\rangle_\pm = \exp\left\{\pm i\omega t - \frac{\alpha^2}{4}(c_2 \pm i c_1)\omega t - \frac{1}{2}\sigma t\right\} .$$

It can be shown that for any conductivity all <a> modes are decreasing with time.

$\Rightarrow$ **no instability** at $t \gg \tau_c$

EVOLUTION OF <ENERGY> AND <HELICITY>

Energy and magnetic helicity $\mathcal{E}_{\boldsymbol{k}\lambda} = \left(\dot{a}^2_{\boldsymbol{k}\lambda} + a^2_{\boldsymbol{k}\lambda} k^2\right) \qquad \mathcal{H}_{\boldsymbol{k}\lambda} = \lambda a^2_{\boldsymbol{k}\lambda}$

Equation for the second moments

$$\frac{d}{dt'}\begin{pmatrix} \langle x^2 \rangle \\ \langle \dot{x}^2 \rangle \\ \langle x\dot{x} \rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ \alpha^2 c_3 & -\alpha^2 c_2 & -2 \\ -1+\alpha^2 c_1 & 1 & -\alpha^2 c_2 \end{pmatrix} \begin{pmatrix} \langle x^2 \rangle \\ \langle \dot{x}^2 \rangle \\ \langle x\dot{x} \rangle \end{pmatrix}$$

$$c_3 = \int_0^\infty \langle \xi(t')\xi(t'-\tau') \rangle \left(1+\cos(2\tau')\right) d\tau'$$

$$\langle \mathcal{E}_{\boldsymbol{k}\lambda} \rangle = \frac{1}{2k} e^{-\sigma t} \left[\langle x^2 \rangle \left(k^2 + \frac{1}{4}\sigma^2 \right) - \langle \dot{x} x \rangle \sigma + \langle \dot{x}^2 \rangle \right] \qquad \langle \mathcal{H}_{\boldsymbol{k}\lambda} \rangle = \lambda e^{-\sigma t} \langle x^2 \rangle \, .$$

Solution:

$$\langle x^2 \rangle = \frac{1}{2}\left[u_0 e^{\nu_0 \omega t} - u_+ \left(e^{\nu_- \omega t} + e^{\nu_+ \omega t} \right) \right] ,$$

$$\langle \dot{x}^2 \rangle = \frac{\omega^2}{2}\left[u_0 e^{\nu_0 \omega t} + u_+ \left(e^{\nu_- \omega t} + e^{\nu_+ \omega t} \right) \right] ,$$

$$\langle x\dot{x} \rangle = \frac{\omega u_+}{2i}\left(e^{\nu_+ \omega t} - e^{\nu_- \omega t} \right)$$

$$\nu_0 = \frac{1}{2}\alpha^2 (c_3 - c_2) \, ,$$

$$\nu_\pm = \pm 2i \left(1 - \alpha^2 \frac{c_1}{4} \right) - \frac{1}{4}\alpha^2 (c_3 + 3c_2)$$

C-COEFFICIENTS

Ornstein-Uhlenbeck process (for illustration): $\langle \xi(t)\xi(t-\tau)\rangle = e^{-\tau/\tau_c}$

$$c_1 = \frac{2(\omega\tau_c)^2}{1+4(\omega\tau_c)^2}\,, \qquad c_2 = \frac{4(\omega\tau_c)^3}{1+4(\omega\tau_c)^2}\,, \qquad c_3 = \frac{\left[2+4(\omega\tau_c)^2\right](\omega\tau_c)}{1+4(\omega\tau_c)^2}$$

$|\delta k| \gg \frac{\sigma}{4}(\Sigma_\chi \tau_c)^2 \quad \Rightarrow$ There are two distinct cases:

1. Good conductor $\quad \sigma \sim T/e^2 \qquad |\omega|\tau_c \gg \mu_5/g^8 T \gg 1 \quad \Rightarrow \quad c_2 \approx c_3 \gg c_1 > 0$

Special case: if spectrum of ξ(t) contains a mode with 2ω there is a *parametric resonance.* For example, if

$\langle \xi(t)\xi(t-\tau)\rangle = e^{-\tau/\tau_c}\cos(2\omega\tau) \quad \Rightarrow \qquad c_1 \to 1/8, \quad c_2 \to -\omega\tau_c/2, \quad c_3 \to \omega\tau_c/2$

2. Poor conductor $\quad \sigma \sim e^2 T$ $\quad$ e.g. QGP near T_c

$\omega\tau_c \gg 1$ $\quad$ for underdamped modes with $k \gg g^4 T$

$|\omega|\tau_c \ll 1$ $\quad$ for underdamped modes with $k \ll g^4 T$ and overdamped modes

$$c_1 \sim (\omega\tau_c)^2\,, \qquad c_2 \sim (\omega\tau_c)^3\,, \qquad c_3 \sim \omega\tau_c$$

STABLE AND UNSTABLE <ENERGY> MODES

$\omega^2 > 0$ if $c_2>0$ (non-resonant) must have $\nu_0\omega > \sigma$

if $c_2<0$ (resonant) must have $\nu_0\omega > \sigma$ or $\mathrm{Re}\,\nu_\pm\omega > \sigma$

👉 never satisfied in a good conductor.

However these conditions can be satisfied in a poor conductor, provided $\sigma\tau_c < 1$

(in poor conductor $\sigma\tau_c \sim e^2/g^4 \ll 1$)

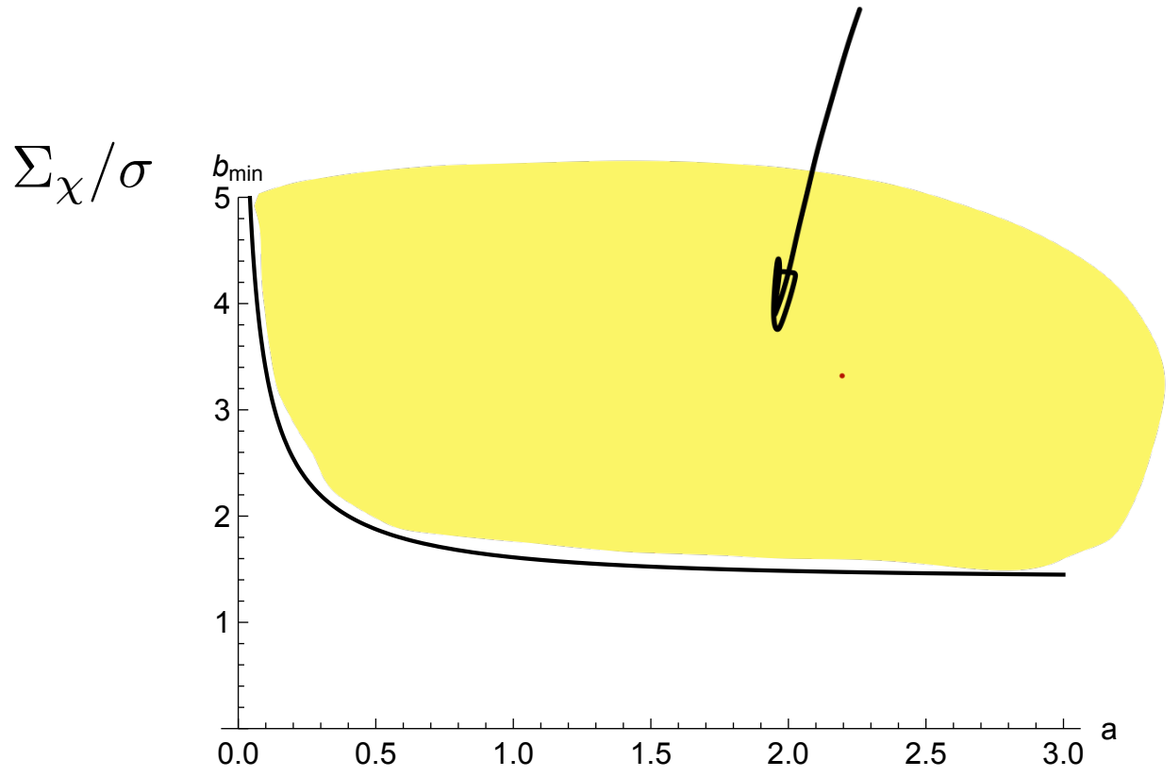
The unstable modes of average energy:

$$\langle \mathcal{E}_{\boldsymbol{k}\lambda} \rangle = \frac{k}{2} u_0 \exp\left\{ \frac{\Sigma_\chi^2}{2k} \frac{2\omega\tau_c}{1+4\omega^2\tau_c^2} t - \sigma t \right\}$$

Does not depend on helicity.

$\omega^2 \leq 0$ There are no unstable overdamped and critical modes.

INSTABILITY REGION



$\sigma\tau_c$

Example: QGP $\quad \sigma \approx 5\,\mathrm{MeV} \quad \tau_c \approx 5\,\mathrm{fm} \quad \Rightarrow$ Instability occurs if $\Sigma_\chi = 15\,\mathrm{MeV}$

Unstable helicity modes: $\quad \langle \mathcal{H}_{\boldsymbol{k}\lambda} \rangle = \dfrac{\lambda u_0}{2} e^{\nu_0 k t - \sigma t} \quad \Rightarrow \quad \langle \mathcal{H} \rangle = 0$

SUMMARY 3

1. I discussed what happens to the chiral instability at $t \gg \tau_c$ by treating the topological charge distribution as spatially homogeneous random process.

2. Field amplitudes $a_{k\lambda}$ in momentum space behave as harmonic oscillators with random frequency. One can obtain ordinary differential equations for $\langle a_{k\lambda} \rangle$, $\langle a_{k\lambda}^2 \rangle$, $\langle a_{k\lambda}^3 \rangle$ etc.

3. Average field amplitudes show no instability.

4. Average energy is unstable in poor conductors (such as QGP) if $\sigma < g^4 T$ and $\Sigma_\chi \gg \sigma$.

5. The magnetic helicity of R and L modes increases exponentially. However, their sum vanishes. Thus, the helicity conservation cannot tame the instability at later times as is does at early times. The instability is not chiral!

